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RÉSUMÉ AND REMARKS ON THE OPEN BOUNDARY CONDITION MINISYMPOSIUM

R. L. SANI

Department of Chemical Engineering, University of Colorado, Boulder, CO 80309-0424, U.S.A.

AND

P. M. GRESHO

Lawrence Livermore National Laboratory, Livermore, CA 94551, U.S.A.

SUMMARY

The incompressible Navier–Stokes equations—and their thermal convection and stratified flow analogue, the Boussinesq equations—possess solutions in bounded domains only when appropriate/legitimate boundary conditions (BCs) are appended at all points on the domain boundary. When the boundary—or, more commonly, a portion of it—is not endowed with a Dirichlet BC, we are faced with selecting what are called open boundary conditions (OBCs), because the fluid may presumably enter or leave the domain through such boundaries. The two minisymposia on OBCs that are summarized in this paper had the objective of finding the best OBCs for a small subset of two-dimensional test problems. This objective, which of course is not really well-defined, was not met (we believe), but the contributions obtained probably raised many more questions/issues than were resolved—notable among them being the advent of a new class of OBCs that we call FBCs (fuzzy boundary conditions).

KEY WORDS Navier–Stokes Boussinesq Boundary condition Open Outflow

INTRODUCTION

Oftentimes one of the biggest difficulties encountered in the mathematical modelling of a physical system is that of boundary conditions. This difficulty arises because of the necessity to truncate the system domain in order to make the problem tractable. Nature is usually silent, or in fact perverse, in not communicating the appropriate ones. In the case of modelling via computational fluid dynamics, the boundary condition dilemma most often occurs at *flow-through* portions of the boundary and in particular at *outflows* or more generally *open* portions of the boundary on which inflow and outflow may even co-exist. The boundary conditions on such open portions of the boundary are a necessary evil that will hereafter be referred to as OBCs. We believe that there are no 'true' OBCs, thus explaining Nature's silence. We also believe and may demonstrate herein that perhaps nowhere else do theory and practice seem to clash so much. This paper and the OBC minisymposium that it describes are based in part on the belief in the premise that OBCs are simply a necessary evil for CFD and on two facts:

1. Computational domains often *must* be truncated from the rest of the universe.
2. Mathematics forces us to select an OBC.

However, mathematics does not tell us how to select OBCs that cause the least upstream

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influence or that permit the most graceful exit. It does not tell us how to synthesize the connection with the rest of the universe.

While we are not able to state the 'best' OBCs, if indeed such exist, we can list some qualities that they would display: they would permit both the flow and anything it carries to exit the domain gracefully and passively and not have any effect on the behaviour of the solution in the domain near the open boundary (and *especially* far from it); they would be transparent; they would lead to the same solution inside the common domain no matter where truncation occurred.

On the other hand, we have Schutt's opinion,[1] perhaps somewhat idealistic but probably shared by many, regarding the placement of the open boundary: 'Nothing interesting should be happening at such a boundary; otherwise the boundary is in the wrong place'.

In order to gain some insight and to attempt to find the *best* OBCs for *incompressible* flows, we organized two OBC minisymposia in conjunction with Professor Cedric Taylor, University of Wales, at his sixth and seventh International Conferences on Numerical Methods in Laminar and Turbulent Flow. A set of four test problems evolved from a solicitation of interest in and test problems for the OBC minisymposia. The final four were characterized by:

1. All were 2D laminar flows.
2. There were two sets of governing equations, isothermal and Boussinesq, with two problems for each set.
3. There were two types of flows with two problems each, steady state and time-dependent.
4. There were two domains for each problem, one nearly long enough and the other definitely (and intentionally) too short—and obtained by simple truncation of the long domain (no extra fine meshes permitted near the outlet).

The governing equations in dimensionless form are

$$\partial \mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + Re^{-1}\nabla^2\mathbf{u} + Fr^{-1}\mathbf{k}T, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{or} \quad \nabla^2 P = \nabla\cdot(Re^{-1}\nabla^2\mathbf{u} + Fr^{-1}\mathbf{k}T - \mathbf{u}\cdot\nabla\mathbf{u}), \tag{2a,b}$$

and

$$\partial T/\partial t + \mathbf{u}\cdot\nabla T = Pe^{-1}\nabla^2 T \tag{3}$$

in primitive variable form. Note that (2b) can only be used to replace (2a) for time-dependent problems; steady state problems need (2a). Note too that boundary conditions for (2b) are usually obtained from (1) *if* the velocity (and acceleration $\partial\mathbf{u}/\partial t$) is *known* on the boundary; at open boundaries $\partial\mathbf{u}/\partial t$ is *not* known, thus requiring pressure BCs to be obtained in other ways.

In the ψ–ω formulation (1) and (2) are replaced by

$$\partial\omega/\partial t + \mathbf{u}\cdot\nabla\omega = Re^{-1}\nabla^2\omega + Fr^{-1}\ \partial T/\partial x, \tag{4}$$

$$\nabla^2\psi + \omega = 0. \tag{5}$$

The solutions of the benchmark problems are available[2–5] and should prove to be useful to other investigators and for reasons other than simply testing OBCs. Here we will reiterate the characteristion of and provide a brief commentary on each test problem.

1. BFS: steady flow past a (simplified) step at Re = 800 (T = 0)

This is not the *true* backward-facing step problem, because we eliminated the inlet region for simplicity so we could focus on the outlet. $Re = 800$ was chosen because it is an interesting value for which a second eddy (separation bubble) exists on the ceiling of the channel. The short domain was purposely selected to 'chop' this top eddy, thus providing the OBC with the challenge of letting fluid also flow *into* the domain.

2. Stratified BFS: test problem 1 except $T \neq 0$, with $Pe = 800$, $Fr = 16/9$

The only change from test problem 1 is to introduce a stably stratified fluid that wants to bounce up and down in a manner somewhat analogous to moutain lee waves. The Froude number selected (with $Ri = 1/Fr = 9/16$) gives an interesting set of five recirculation zones (eddies), three on the floor and two on the ceiling. Also, the additional dimensionless parameters—in addition to the Richardson number above—are: $Gr = Re^2/Fr = 3{\cdot}6 \times 10^5$ and $Ra = PrGr = 3{\cdot}6 \times 10^5$. Here the truncated domain again chops one of the eddies and completely loses the most downstream of them.

3. VSCC: vortex shedding past a circular cylinder at $Re = 100$

The Karman vortex street at $Re = 100$ is a well-studied problem in both physical and computational laboratories. Letting vortices leave the grid passively (and with minimally distorted shape) is the desired behaviour here—but only part of it. Good OBCs will not defile the usual global quantities such as the Strouhal number, lift and drag coefficients, etc. The truncated mesh in this case is especially severe, occurring only four diameters from the cylinder centre and thus in a region in which the vortices are actually still in the formative stage.

4. PBCF: Poiseuille–Benard channel flow at $Re = 10$, $Pe = 20/3$, $Fr = 1/150$

This channel flow would be a simple Poiseuille flow if there were no heating from below and Benard flow (at $Ra = PeRe/Fr = 10^4$) if the ends were closed. However, the heating causes Benard roll cells even in the presence of forced flow—they are translating periodic roll cells. As for vortex shedding, a good OBC will permit a passive exit and not corrupt upstream behaviour, including the quantitative measures of wavelength, frequency, velocity extremes, average Nusselt number, etc.

Following the second minisymposium held at Stanford University at the seventh International Conference on Numerical Methods in Laminar and Turbulent Flows in July 1991, there were nine groups who reported on the performance of the outflow boundary conditions implemented in their particular numerical scheme on the benchmark problems. Unfortunately, none of the groups reported on all the benchmark problems and there was little uniformity in the type of information reported; thus the scope of this summary will be limited by these facts. Below we comment briefly on the general techniques and results of the various groups and, in closing, make some general remarks which we hope will be useful and provide a stimulus for further discussion and investigation of the OBC problem. An earlier, preliminary summary of the 1991 symposium, including some additional discussion, is available in Reference 6.

THEORY

In this section we first attempt to summarize what we (the authors) know (or believe we know) theoretically regarding some 'open' boundary conditions for the incompressible Navier–Stokes equations—partly to set the stage for some of the confusion that follows, but partly, hopefully, to help clear the air. After discussing the PDE case, we summarize some related issues for the spatially discretized case.

The first theoretical notion worth mentioning is this: $\nabla \cdot \mathbf{u} = 0$ on Γ (as well as of course in

Ω) is of crucial importance in incompressible flow and can in some general sense be construed as a boundary condition (both on the OBC portion as well as on all other portions).

Next we summarise three well-posed OBCs for all quantities (symbolized by φ) *except* the normal component of velocity:

(i) $\varphi = 0$ (or other specified value) is usually a poor choice, especially in practice (i.e. via approximate/numerical solutions), because the thin boundary layer that it endenders for large Peclet or Reynolds number (the case of most interest) can cause havoc with numerical schemes in the form of wiggles (oscillations) that permeate far upstream if the 'artificial' outflow boundary layer is not resolved.[7]

(ii) $\partial\varphi/\partial n = 0$ is usually a good choice—the thin boundary layer that exists is usually innocuous even when not resolved and is an example of what some call a good feature of a numerical method (ignoring boundary layers that it cannot see rather than 'overreacting' to them).[7]

Related to (i) and (ii) above is the statement of Naughton:[8] 'From the theory of singular perturbation problems it is well-known that boundary layers are weaker if boundary conditions on the derivatives are used (*soft* boundary conditions) rather than boundary conditions on the function itself (*hard* boundary conditions)'. In both (i) and (ii) the boundary layer thickness is $O(Re^{-1})$ or $O(Pe^{-1})$ and is the region in which advection and diffusion are *both* significant/important.

(iii) $\partial\varphi/\partial t + V\partial\varphi/\partial n = 0$, where V is 'user-specified', but cannot be zero, and should be positive if fluid is leaving the domain. (The average normal velocity through the boundary is a reasonable candidate.) This OBC may be gaining in popularity over $\partial\varphi/\partial n = 0$ for reasons which are not entirely clear to us.

Turning now to what might be accurately be called the *nemesis* of incompressible flow, we summarize a portion of what we know about OBCs for the *normal* momentum equation: here $u_n = \mathbf{n}\cdot\mathbf{u}$ and $\partial u_n/\partial n = \mathbf{n}\cdot\nabla(\mathbf{n}\cdot\mathbf{u})$, at least for straight/planar boundaries.

(a) $\partial u_n/\partial n = 0$ is illegal ('hopelessly ill-posed'—V. Girault, personal communication) because of insufficient information; the system (PDE's and boundary conditions) is under determined, resulting (in general) in an infinite number of solutions. In addition, in 2D it is often overly restrictive on u_τ (the tangential component of velocity), causing $u_\tau = 0$ (via $\nabla\cdot\mathbf{u} = 0$) regardless of the *actual* (or ostensible) boundary condition applied to u_τ.

(b) $\partial u_n/\partial n = 0$ and $P = 0$ is illegal (ill-posed) because of too much information; the system is overdetermined and no solution exists in general.

(c) $\partial u_n/\partial n = 0$ and $P = 0$ at just *one* point (usually taken to lie on Γ_{OBC}) may be legal, but we have our doubts because it seems to be lacking a boundary condition for the pressure Poisson equation (PPE) (2b)—as is also the case with (a).

(d) $\partial u_n/\partial t + V\partial u_n/\partial n = 0$ is illegal (ill-posed) like (a) above owing to insufficient information. Unlike (a), however, it does not cause $u_\tau = 0$.

(e) $\beta Re^{-1}\,\partial u_n/\partial n - P = 0$ (or other specified value), where $\beta = 1$ (conventional form) or $\beta = 2$ (stress divergence form), is well-posed but may not always be useful. Note that $\beta = 1$ is to be preferred because the stress divergence formulation *requires* $\partial u_n/\partial\tau + \partial u_\tau/\partial n = 0$ as the homogenous OBC for u_τ (zero shear stress), which is often worse than $\partial u_\tau/\partial n = 0$ that obtains when $\beta = 1$ (see e.g. Reference 9). Note too that this BC is an NBC (natural boundary condition) when implemented via the Galerkin weak formulation.

(f) $\partial u_n/\partial t + V\partial u_n/\partial n = ReVP$ is, we believe (assert), legal (well-posed) but, to our knowledge, untested. It brings the pressure back into (d) à la (e).

To *show* the ill-posedness of OBCs (a), (c) and (d), consider a simple *counterexample*: the steady Stokes equations in a domain comprising the unit square with $u = v = 0$ top and bottom and OBC's elsewhere. The *family* of solutions $u = a(y^2 - 1)$, $v = 0$, $P = 2ax + b$ for *all* a and b

satisfies the OBC's of (a) and (d), and the family comprising the same u, v but with $P(x) = P_0 + 2a(x - x_0)$ for arbitrary a satisfies (c) and reduces the 'indeterminateness' from a double to a single infinity of parameters. In contrast, (e) and (f) above yield the *unique* solution $\mathbf{u} = 0$, $P = 0$. The way to prove that (b) is ill-posed[10] is to compare it with (e) for $\beta = 1$, which is known to be well-posed; since the latter does not generally display $\partial u_n/\partial n = 0$ and $P = 0$, (b) is overspecified and thus ill-posed.

In the spatially discretized approximation the PDE's simplify to DAE's (differential–algebraic equations) in the time-dependent case and to non-linear algebraic equations in the steady case. We now attempt a brief summary of some of the related OBC issues—for additional discussion see Reference 11. The vehicle for the 'demonstration' will be the simple steady Stokes equations, whose discrete form reads

$$Ku + Gp = f, \tag{6a}$$

$$Du = g, \tag{6b}$$

where $Ku \approx -Re^{-1}\nabla^2\mathbf{u}$, $Gp \approx \nabla P$, $Du \approx -\nabla\cdot\mathbf{u}$ and f and g are data that include inhomogeneous Dirichlet boundary conditions and, in f, any 'body force' terms. In discrete simulations which are said to utilize OBC (a) or (c) above, the gradient matrix G always annihilates a constant (hydrostatic, p_h) pressure vector, e.g. $G\cdot(1 \rightarrow 1)^T = 0$, which is simply a statement that the hydrostatic pressure vector is always in the null space of the (singular) Stokes matrix

$$A = \begin{bmatrix} K & G \\ D & 0 \end{bmatrix}, \tag{7}$$

i.e.

$$A\begin{bmatrix} 0 \\ p_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $p_h \equiv (1 \rightarrow 1)^T$. Now in some finite difference (or finite volume) codes it is true that D and G are *almost* transposes of each other; in fact, except at or near Γ_{OBC}, it is often true that $D^T = G$. One more very relevant and very important point is this: even though $Gp_h = 0$, there are *no redundant continuity equations* in (6b), so that when one removes the singularity by specifying the pressure at one node, one also sacrifices both local (at the selected node) and global discrete mass conservation. On the other hand, if one does not 'peg a pressure', thus retaining mass consistency, the singular A-matrix generally leads to an ill-posed algebraic system—(6) has no solution for general data (f, g). The *only* time the unpegged case is solvable is when the right-hand side is orthogonal to the null vector of A^T, i.e., the *solvability condition* on (6) is then

$$w^T f + q^T g = 0, \tag{8}$$

where $(w, q)^T$ is the null vector of A^T:

$$Kw + D^T q = 0 \tag{9a}$$

$$G^T w = 0. \tag{9b}$$

(To obtain (8), take the inner product of $(w, q)^T$ with (6), using (9).)

If an FDMer were sufficiently bothered by the above problems (peg a pressure and lose mass balance; do not and come up against an ill-posed problem) to ask an FEMer for help, the response might be the following. Yes, I can tell you how to generate a solvable problem and retain mass consistency, but it will involve *changing the data* (f). It goes like so and relies on the fact that $G^T = D$ 'almost everywhere' (away from the open boundary).

(i) Solve the problem

$$Ku + D^T p = f, \tag{10a}$$

$$Du = g \tag{10b}$$

for your given data. Do not specify (peg) a pressure. It turns out that the (symmetric) matrix in (10) is non-singular ($D^T p_h \neq 0$ in particular) and thus the problem in linear algebra is well-posed—for *all* (f, g)—as are the underlying PDEs, with OBC(e), which (10) approximates.

(ii) Set up the *singular* system

$$Ku + Gp = f + (G - D^T)p \equiv \hat{f}, \tag{11a}$$

$$Du = g, \tag{11b}$$

a system whose (unsymmetric) matrix is singular ($GP_h = 0$) but whose modified right-hand side $(\hat{f}, g)$ is now *consistent by construction*, i.e. the solution to (11) is the *same* as that of (10)—at least up to p_h: $p + \gamma p_h$ for arbitrary γ also solves (11). There are four key points:

1. Discrete mass conservation is achieved.
2. The system is *consistent singular*.
3. The implied OBC for the PDE may be of the 'fuzzy' variety.
4. The consistent data for (11) can ostensibly only be generated by first solving (10).

It is the last point that is the 'awkward' one (although the third point is also not pleasant) and the one that makes the FDM system (and solution) look more like that which would have come from the analogous FEM in the appropriate weak formulation, where here 'appropriate' means that the pressure gradient term was integrated by parts. This causes D^T to *look like* a gradient everywhere except on Γ_{OBC}, where it looks more like a pressure *force* on the boundary. That is, 'since (11) has the same solution (up to p_h) as (10), just solve (10)' seems to be the bottom line.

There is, however, a flip-side to this construction: (e) looks too much like (and would respond like) the FEM system that can generate bad results near Γ_{OBC} when there is a significant *body force* (e.g. buoyancy, centrifugal force) which must be balanced by the (largely hydrostatic) pressure distribution, as in 'stratified' flow (test problems 2 and 4). For further elaboration of these issues see References 12 and 13, but the basic problem is easy to state: when large variations in pressure exist at the outlet, BC (e) will usually cause large errors at and near the outlet—but it would, assert, give excellent results for test problems 1 and 3.

Thus we now know how to make FDM mimic FEM, which may sometimes be useful. However, we still do not known enough about going in the other direction; i.e. telling the FEMer how to emulate the $\partial u_n/\partial n = 0$ OBC by cleverly pegging P to generate a non-singular (reduced) matrix yet generating acceptable, or even very small, mass inconsistencies—unless it is simply this (which we have briefly experimented with and sometimes obtained disappointing results[14]): do not integrate ∇P by parts when generating the weak formulation, an easy thing to do—usually. (It is *not* easy if the simplest finite element, that using piecewise-constant pressure, is employed, since then the G-matrix is zero.)

Finally, for FDM or FEM when G is a gradient everywhere (and p_h is thus a mode), we are confronted with the disturbing question, related to the alleged ill-posedness of the PDE's. 'What then is the (implied) boundary condition for the (implied) pressure Poisson equation?'. This is the situation that is often encountered in FDM and less often in FEM—occurring only when ∇P is not integrated by parts.

SUMMARY OF OBC CONTRIBUTIONS

The résumé of the contributions will be grouped relative to the spatial discretization method; i.e. finite difference, finite element, or spectral element, and unfortunately will contain a depth of detail which varies greatly owing in part to the contributions provided by the various contributors. Also, we admit up front that this summary is written by FEM guys who believe that they pretty well understand FEM OBC's—both virtues and maladies—but they do not much understand FDM OBC's, a fact that must be kept in mind when reading this paper.

Finite difference

I. *A Bottaro*, IMHEF-EPFL, Lausanne, Switzerland

1. Method. Transient, primitive variable on a staggered mesh.

2. Test problem. PBCF.

3. OBC. Four OBCs were tested.

(a) OBC1: weighted true upwinding

Here (1) without the ∇P-term and (3) are applied at the exit utilizing forward Euler in time, backward differencing in space and a standard upwind differencing scheme on the advection terms. After predicting the normal component of velocity at the new time level, it is adjusted via a constant additive factor to satisfy mass conservation.

(b) OBC2: true upwinding replaced by zero normal derivative on inflow portions

Here method (a) is applied everywhere except on those portions on which there is *inflow*, i.e. $u_n < 0$, where $\partial\phi/\partial n = 0$, $\phi = (u, v, T)$, replaces it.

(c) OBC3: average convective derivative

Here

$$\partial\phi/\partial t + c\,\partial\phi/\partial n = 0, \tag{12}$$

where $\phi = (u, v, T)$ and c is a 'constant average phase speed' applied everywhere on Γ_{OBC}.

(d) OBC4: convective derivative

This is the same as OBC3 except that c is replaced by the local normal velocity, i.e. u_n.

4. Results. OBC1 leads to distorted convective cells which are confined primarily to less than one wavelength from the outlet. OBC2 leads to a gross distortion in the flow and temperature fields. OBC3 distorts the convective cells locally near the exit like OBC1, a result of a large pressure build-up near the exit which appears to be related to the choice of the value of the phase speed c. OBC4 diverges when a large enough portion ($\geq 30\%$) of the open boundary has $u_n < 0$. The author assessed the 'practical' performance of all four conditions as approximately the same, with OBC3 offering the simplest implementation. Figure 1 illustrates OBC3 with the imposed speed to equal 1.0 (the average flow velocity), the best of the three reported results for imposed speeds of 0·6, 0·8 and 1·0. (The average channel velocity is 1·0.)

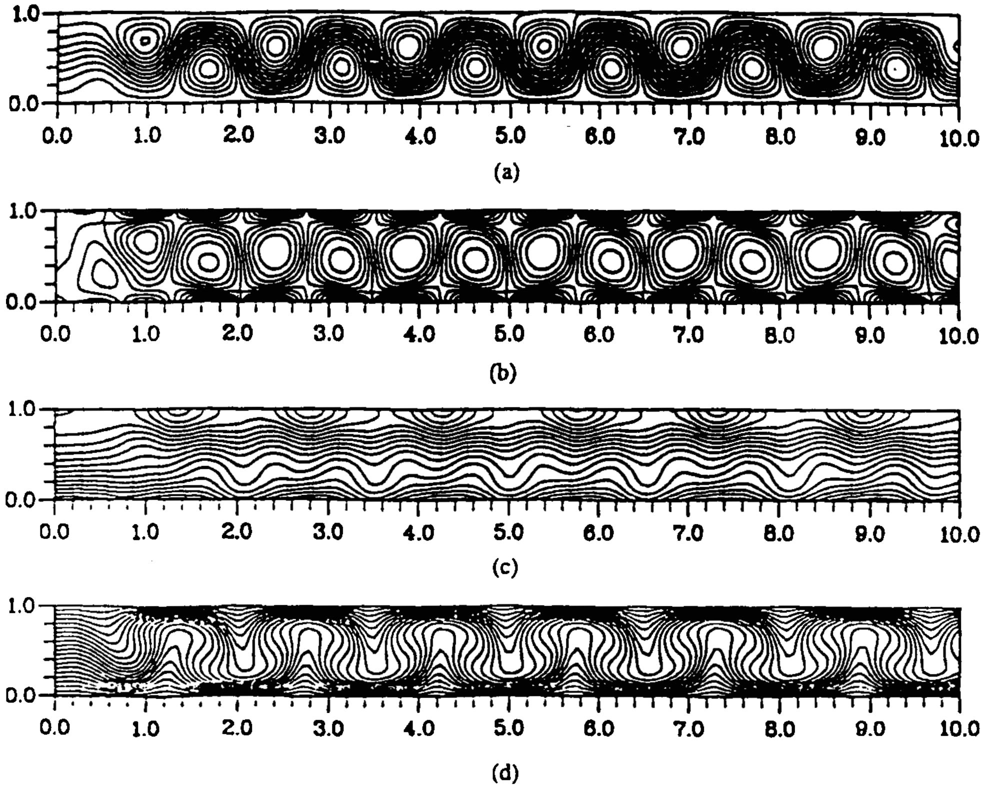

Figure 1. Poiseuille–Benard channel flow: (a) streamlines; (b) vorticity; (c) pressure; (d) temperature

II. M. H. Kobayashi, J. C. F. Pereira and J. M. M. Sousa, Mechanical Engineering Department, Institute Superior Tecnico, Lisbon, Portugal

1. Method. Transient, primitive variable on both staggered and non-staggered meshes.

2. Test problems. BFS, SBFS and PBCF.

3. OBC. Five OBCs were tested; in each case the boundary condition requires iterating at each time step.

(a) OBC1

One-sided difference approximation to a zero first derivative in the main flow direction; staggered mesh. At each instant in time

$$\phi^{k+1}_{NI,J} = \phi^{k}_{NIM1,J}, \tag{13}$$

where $\phi = (U, V, T)$ and the superscript is an iteration index.

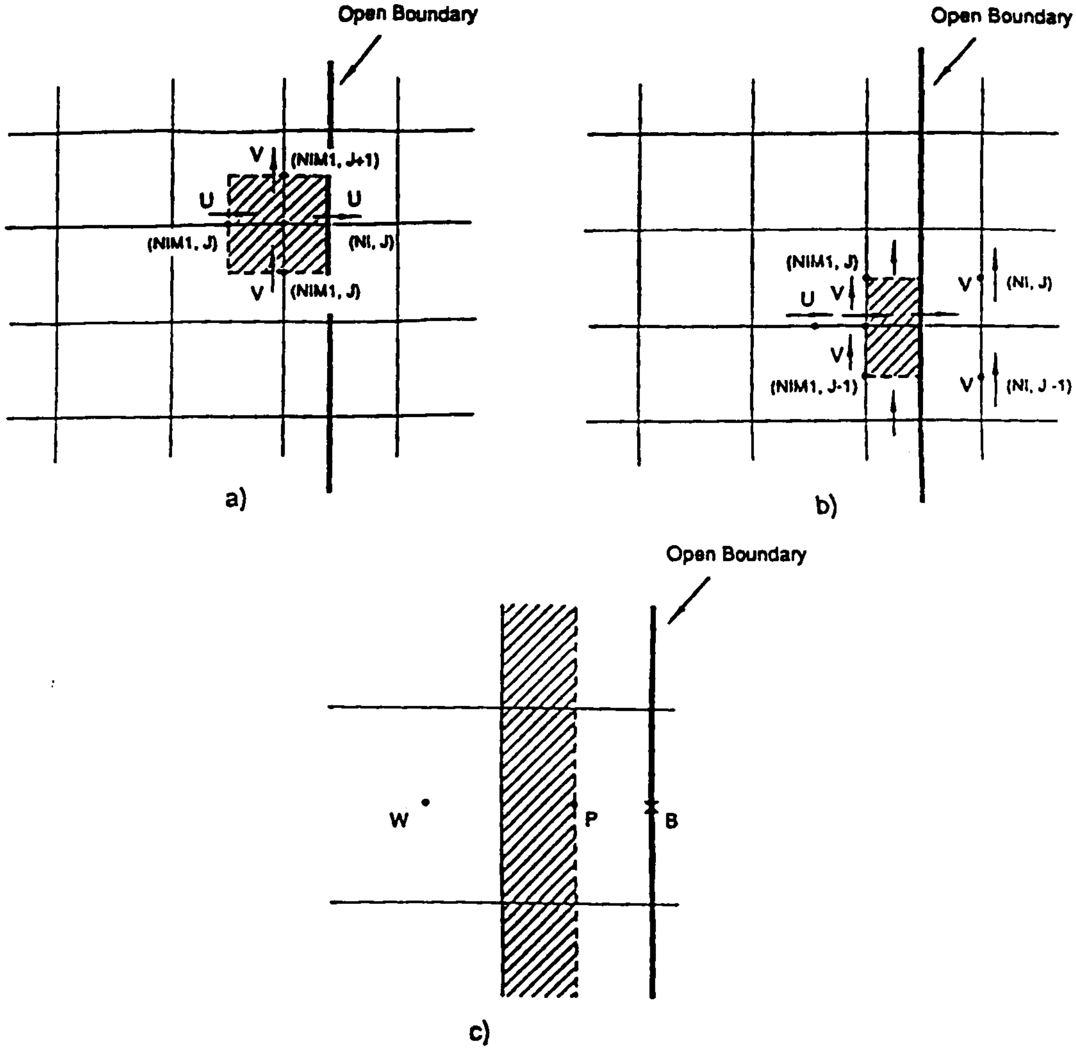

Figure 2. Mesh details (continuity equation is applied to shaded area only): (a) staggered grid, U-momentum equation; (b) staggered grid, V-momentum equation; (c) non-staggered grid

(b) OBC2

At each instant in time the U-velocity is obtained by satisfying mass conservation on the control volume using one-sided differences (see Figure 2(a)). In a similar fashion the V-velocity is evaluated by mass balance applied to half the control volume (see Figure 2(b)); staggered mesh.

$$U^{k+1}_{NI,J} = [(V^{k}_{NIM1,J} - V^{k}_{NIM1,J+1})\Delta x + \Delta y U^{k}_{NIM1,J}](\Delta y)^{-1}, \tag{14}$$

$$(\Delta x/2)V^{k+1}_{NI,J} - (\Delta x/2)V^{k}_{NIM1,J-1} - (\Delta x/2)\tfrac{1}{4}V^{k+1}_{NI,J-1} + U^{k}_{NI,J-1}\Delta y - \tfrac{1}{2}U^{k}_{NI,J-1}\Delta y - \tfrac{1}{2}U^{k}_{NIM1,J-1}\Delta y + (\Delta x/2)\tfrac{3}{4}V^{k}_{NIM1,J} = 0. \tag{15}$$

(The value of $V^{k+1}_{NI,J}$ may be evaluated for $J = 3$, $NJM1$, since a solid wall exists at $J = 2$.) For the temperature field

$$T^{k+1}_{NI,J} = T^{k}_{NIM1,J}. \tag{16}$$

(c) OBC3

A monochromatic travelling wave was assumed at the outlet, with a prescribed speed equal to the mean channel speed; staggered mesh.

$$\phi^{k+1}_{NI,J} = \phi^{k}_{NI,J} - (\bar{u}\Delta t/\Delta x)(\phi^{k}_{NI,J} - \phi^{k}_{NIM1,J}). \tag{17}$$

(d) OBC4

Same as OBC2 but with a linear extrapolation for the temperature field from two interior nodes, i.e.

$$T^{k+1}_{NI,J} = 2T^{k}_{NIM1,J} - T^{k}_{NIM2,J}, \tag{18}$$

and a uniform mesh.

(e) OBC5

Linear extrapolation for outflow pressure field and velocity, e.g.

$$P_{\mathrm{B}} = P_{\mathrm{P}} + (P_{\mathrm{P}} - P_{\mathrm{W}})(1 - fx_{\mathrm{W}}),$$

where fx_W is a 'linear interpolation factor, i.e. if this grid face is located midway between adjacent nodes W and P, then $fx_{\mathrm{W}} = 0{\cdot}5$' (see Figure 2(c)).

4. Results

(a) BFS

OBC3 treatment reverts to zero first derivative (OBC1) when steady state is reached. No significant differences were found in the steady state solutions obtained using OBC1 and OBC3 with the open boundary located at $x = 7$ or 15, except for the V-velocity for the case of OBC located at $x = 7$.

(b) SBFS

OBC5 treatment yields close agreement of the present numerical predictions with reference values for both short and long domains, except for the V-velocity component at $x = 3$. However, this discrepancy represented less than 5% of the reference velocity and occurred in a section far from the outflow boundary. The proposed OBC is reported to be an accurate procedure to be used with non-staggered grids.

(c) PBCF

OBC1 proved to be a wrong and inaccurate open boundary condition in the case of the presence of travelling waves, with OBC effects extending far upstream. OBC2 yielded satisfactory 'outside domain U- and V-values'; this condition is independent of travelling waves. OBC3 proved to be the best numerical treatment for velocity and temperature fields. However, the phase speed of the wave, taken as the mean channel velocity, was about 10% in error. Linear extrapolation in the temperature field at the boundary, OBC4, did not yield improved results compared with OBC2, because this field displayed a minimum at OBC location at time t_τ. Figure 3 displays some results.

III. *E. Gurgey*, Hermann-Fottinger–Institut, Berlin, Germany, and *F. Thiele*, DLR, Abt. Turlrilenzorschum, Berlin, Germany

1. Method. Steady state method employing fourth-order streamfunction equation.

2. Test Problem. BFS.

3. OBC. For the fourth-order equation two OBCs are needed and the ones employed were

$$(\nabla \times \psi)\cdot\nabla\nabla^2\psi = 0, \qquad (\mathbf{u}_\infty\cdot\nabla)\nabla\times\psi = 0, \tag{19}$$

where $\mathbf{u}_\infty$ is specified. (The authors state that $\mathbf{u}_\infty$ was set equal to the undisturbed flow.)

4. Results. The streamfunctions calculated for a long domain ($L = 15$) and a short domain ($L = 7$) were in excellent agreement. A comparison of the results with the benchmark solution is shown in Figure 4. The agreement between the short domain ($L = 7$) and the same location in the long domain ($L = 15$) as well as with the benchmark problem is very good. However, in contrast with the other contributions, a graded mesh was used at the open boundary.

IV. *J. Goodrich*, NASA Lewis Research Center, Cleveland, OH, U.S.A., *and T. Hagstrom*, University of New Mexico, Albuquerque, NM, U.S.A.

1. Method. Transient, streamfunction–vorticity formulation; second-order, centred in space with second-order Adams–Bashford discretization of transport terms and Crank–Nicolson discretization of viscous terms in time; uniform mesh.

2. Test problem. BFS.

3. OBC. The property that the spatial decay to Poiseuille flow for moderate values of Reynolds number Re scales asymptotically as Re^{-1} is used to derive two boundary conditions of the form $B_j(\psi - \psi_\infty) = O(Re^{-2})$, where $\psi_\infty(y)$ is the Poiseuille flow streamfunction.

(a) OBC1

$$B_1 = \partial^2/\partial x^2. \tag{20}$$

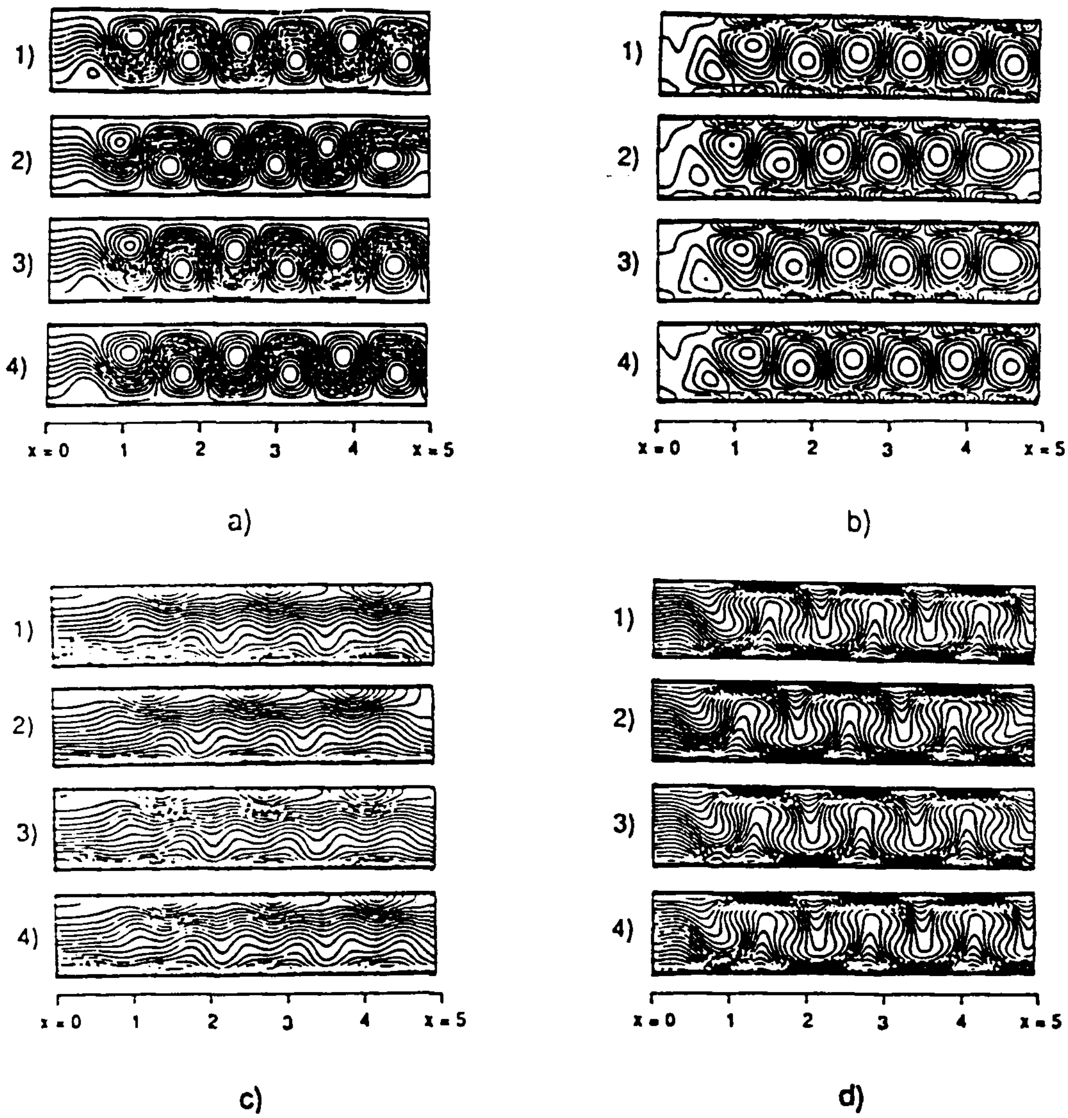

Figure 3. PBCF field plots at $t = t_\tau$ for (1) $x_{max} = 10$ and (2)–(4) $x = 5$ using (2) OBC1, (3) OBC2 and (4) OBC3: (a) streamlines; (b) vorticity contours; (c) pressure contours; (d) temperature contours

(b) OBC2

$$B_2 = \prod_{i=1}^{2} \left(\frac{\partial}{\partial x} - \frac{\lambda_i(0)}{Re} - \frac{\mathrm{d}\lambda_i}{\mathrm{d}s}(0)\frac{\partial}{\partial t} \right), \tag{21}$$

where λ_i is obtained from the approximate, reduced spatial Orr-Sommerfeld equation

$$s + \lambda_i u_\infty(y) W_i'' - \lambda_i u_\infty''(y) W_i = W_i^{(\mathrm{iv})}, \tag{22}$$

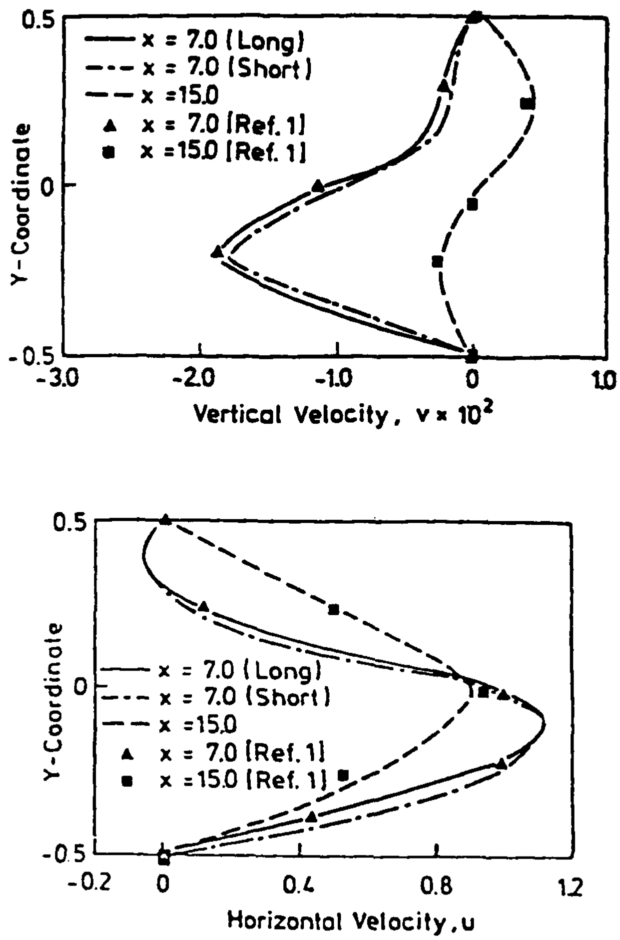

Figure 4. Velocity profiles for backward-facing step using a fourth-order streamfunction equation

where s is the temporal exponential factor, i.e. $\exp(st)$, and λ_i is an x-spatial Fourier transform variable. The smallest eigenvalue $\lambda_i(0)$ at $s = 0$ is used in OBC2. (See Reference 15 for more details.)

4. Results. Both boundary conditions worked very well. For both eddies none of the key parameters changed by more than 3% as the boundary was moved from $L = 25$ to 7. Table I displays the results and a comparison with the benchmark solution[2] for the BFS.

V. *K. Srinivasan and S. C. Rubin*, Department of Aerospace Engineering and Engineering Mechanics, University of Cincinnati, Cincinnati, OH, U.S.A.

Table I

Bnd. op.	Boundary location	Vortex centre	ψ_{centre}	ω_{centre}	Eddy length
		Lower eddy			
Reference case	30	(3·35, 0·3)	−0·0342	2·283	6·10
2	25	(3·359, 0·2969)	−0·034166	2·268	6·042
1	15	(3·359, 0·2969)	−0·034167	2·267	6·033
2	15	(3·359, 0·2969)	−0·034166	2·269	6·035
1	7	(3·328, 0·2969)	−0·034119	2·285	5·860
2	7	(3·328, 0·2969)	−0·034125	2·278	5·888

Bnd. op.	Boundary location	Vortex centre	ψ_{centre}	ω_{centre}	Sepuration point
		Upper eddy			
Reference case	30	(7·4, 0·8)	0·5064	−1·322	4·85
2	25	(7·422, 0·8125)	0·5069	−1·189	4·799
1	15	(7·422, 0·8125)	0·5068	−1·190	4·803
2	15	(7·422, 0·8125)	0·5069	−1·188	4·793
1	7	(7, 0·7813)	0·5080	−1·145	4·636
2	7	(7, 0·7969)	0·5078	−1·013	4·662

1. Method. Steady, primitive variable formulation of the reduced (parabolized) Navier–Stokes equations.

2. Test problem. BFS.

3. OBC. Dirichlet boundary condition for pressure if $u_n > 0$. The negative convective fluxes are neglected, eliminating an OBC for velocity. A flux-vector-splitting technique is used for the pressure field such that the value p_i is located between the $(i-1)$th and ith grid locations.

4. Results. The comparison with the BFS benchmark problem is very good for both the long and short domains. Figure 5 displays a comparison of streamwise velocity profiles with the benchmark solution, while Table II gives a comparison of the corresponding eddy lengths. In all cases there is good agreement between the results for various truncated domains as well as with the benchmark solution.

VI. *A. O. Demuren and R. V. Wilson*, Old Dominion University, Norfolk, VA, U.S.A., *and T. Hagstrom*, University of New Mexico, Albuquerque, NM, U.S.A.

1. Method. Steady, primitive variable formulation with a TEACH-type discretization with staggered mesh.

2. Test problem. BFS.

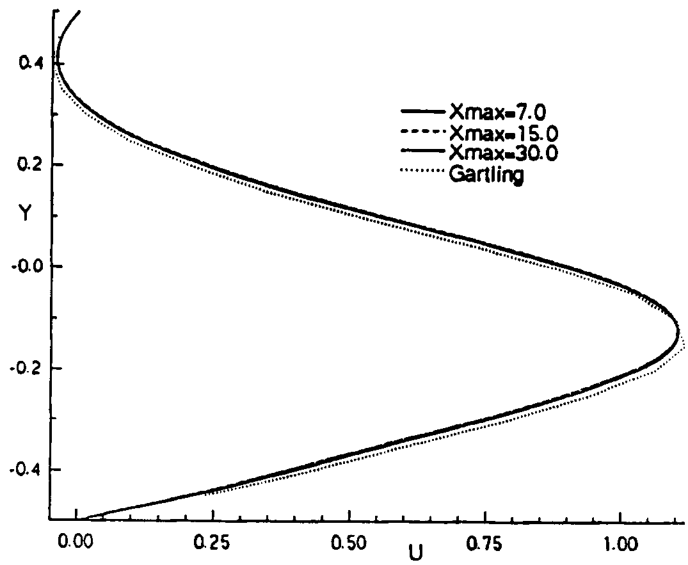

Figure 5. Velocity profiles for backward facing step using parabolized Navier–Stokes equations

Table II

Domain length	Lower eddy reattachment	Upper eddy separation	Upper eddy reattachment
$x_{max} = 7$	6·15	5·09	—
$x_{max} = 15$	6·22	5·125	10·22
$x_{max} = 30$	6·22	5·09	10·25
Gartling[2]	6·10	4·85	10·48

3. OBC. Three different OBCs were tested.

(a) OBC1

$$\partial \mathbf{u}/\partial n = \mathbf{0}. \tag{23}$$

(b) OBC2

$$-P + Re^{-1}\,\partial u_n/\partial n = 0, \tag{24a}$$

$$\partial u_\tau/\partial n = 0. \tag{24b}$$

(c) OBC3

$$\partial^2 u_n/\partial n^2 = 0, \tag{25a}$$

$$\partial^2 u_\tau/\partial n^2 = 0. \tag{25b}$$

4. *Results.* OBC1 produced the best results in the sense of minimal sensitivity to calculation length. Recirculation length, strength and location of the first eddy varied by less than 1% for lengths of 7, 10, 15 and 30. It is also the only boundary condition which produced the correct pressure distribution across the outflow for the $L = 7$ computation. OBC2 and OBC3 produced deviations of about 2% between calculations with $L = 7$ and the longer domain. They also produced nearly uniform pressure distribution at the outflow, which is incorrect. The authors conclude that OBC1 appears best for this problem probably because the streamlines at outflow are nearly parallel for all lengths.

Finite element

VII. *P. L. Betts and A. I. Sayma*, Department of Mechanical Engineering, UMIST, Manchester, U.K.

1. *Method.* Transient, primitive variable formulation. Galerkin spatial discretization method using four-node bilinear isoparametric quadrilaterals for velocity and piecewise-constant pressure; forward Euler time-marching scheme with a fractional step pressure correction method.

2. *Test problem.* BFS.

3. *OBC.* Five different OBCs were tested for the short domain ($L = 7$).

(a) OBC1

$$-P + Re^{-1}\,\partial u_n/\partial n = 0, \tag{26a}$$

$$Re^{-1}\,\partial u_\tau/\partial n = 0. \tag{26b}$$

(b) OBC2

$$-P + Re^{-1}\,\partial u_n/\partial n = 0, \tag{27a}$$

$$u_\tau = 0. \tag{27b}$$

(c) OBC3

$$-P + Re^{-1}\,\partial u_n/\partial n = \bar{P}, \tag{28a}$$

$$Re^{-1}\,\partial u_\tau/\partial n = 0, \tag{28b}$$

where $\bar{P}$ was the pressure at $x = 7$ computed from the $L = 15$ solution by averaging the values of pressure from elements adjacent to $x = 7$.

(d) OBC4

$$-P + Re^{-1}\,\partial u_n/\partial n = f_1, \tag{29a}$$

$$Re^{-1}\,\partial u_\tau/\partial n = f_2, \tag{29b}$$

where f_1 and f_2 were obtained by averaging the values obtained from the $L = 15$ solution at $x = 7$.

(e) OBC5

In this case the functions f_1 and f_2 were replaced by using values computed using results in two elements preceding the boundary at the previous time step.

4. *Results*

(a) OBC1

There were only small differences between the long-domain ($L = 15$) and short-domain ($L = 7$) results, except that in the latter the pressure near the exit appeared to have a flatter profile.

(b) OBC2

The results were virtually the same as for OBC1; the tangenial component of velocity at the exit computed via OBC1 was very small compared with the normal component of velocity and hence in this case setting it to zero leads to only a small perturbation from the solution.

(c) OBC3

Here the normal velocity profile ($L = 7$) coincided with the long-domain ($L = 15$) results to graphical accuracy and the pressure variation was also nearly identical.

(d) OBC4

In this case the solution started to *wiggle* and then diverged. When the tangential traction condition was set equal to zero instead of f_2, the solution obtained was equivalent to that for the OBC3 case. The latter occurs because the magnitude of the tangential traction condition $Re^{-1}\,\partial u_\tau/\partial n$ is four orders smaller than the value of the pressure at the exit.

(e) OBC5

In this case the solution diverged (on the short domain) even when the tangential traction was set equal to zero. When there is no inflow (e.g. long domain), the method is often successful.

A résumé of eddy lengths and vortex centre locations is given in Table III.

Remark. An OBC similar to OBC5, in which f_1 and f_2 are updated at each iteration (or time step for a transient simulation), was advocated and (usually) successfully implemented by Taylor *et al.*[16]—but again not for OBC's with inflow.

Table III

	Lower eddy					Upper eddy				
Domain Mesh	$L = 15H$ (90 × 20)	$L = 7H$ (42 × 20)				$L = 15H$ (90 × 20)	$L = 7H$ (42 × 20)			
Team		OBC1	OBC2	OBC3	OBC4		OBC1	OBC2	OBC3	OBC4
Vortex centre	(2·92, 0·175)	(2·90, 0·175)	(2·90, 0·175)	(2·90, 0·175)	(2·90, 0·175)	(6·92, 0·283)				
V at vortex centre	−0·0313	−0·0314	−0·0314	−0·0314	−0·0314	0·502				
Vorticity at vortex centre	−2·472	−2·460	−2·460	−2·445	−2·445	1·536				
Reattachment at $x =$	5·606	5·578	5·578	5·592	5·592	10·43				
Separation at $x =$						4·20	4·29	4·29	4·28	4·28
Recirculation length						6·23				

VIII. *T. C. Papanastasiou*, Department of Chemical Engineering, The University of Michigan, Ann Arbor, MI, U.S.A.

1. Method. Steady, primitive variable.

2. Test problems. BFS and SBFS.

3. OBC. In generating the Galerkin weak form of the equations for C^0 velocity and C^{-1} pressure finite element basis functions, the viscous stress and pressure gradient terms in the Navier–Stokes equations must be *integrated by parts* using Green relationships so that the Galerkin projection will be convergent. Usually the resulting boundary integrals are utilized to form general *natural boundary* conditions as, for example, employed in the preceding contribution of Betts and Sayma. However, the technique employed here is to retain these boundary integrals in the weak form on the 'open' portions of the boundary, i.e. where velocities are not specified. Neglecting discretization error, this is equivalent to not *imposing* any boundary condition specifically as an OBC.

4. Results.

The comparison of the long- and short-domain results with the BFS and SBFS benchmark solutions[2,3] was very good.[17] In the case of the SBFS the author notes that a refinement of the mesh did not significantly change the velocity and temperature traces on the open boundary ($x = 7$), suggesting a mesh-converged result or insensitivity to mesh refinement. Specifying a pressure datum at a unique outflow node appears to improve convergence for non-linear problems.

Spectral element

IX. *A. G. Tomboulides, M. Israeli and G. E. Karniadakis*, Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ, U.S.A.

1. Method. Transient, primitive variable, spectral element.

2. Test problems. BFS and VSCC.

3. OBC. The OBC here was velocity boundary conditions employing viscous sponges and a parabolized equation of motion at the *outflow* with variable viscosity in the streamwise and transverse directions. The streamwise viscosity is reduced to zero in a sponge layer close to the outflow, leading to a parabolized form of the Navier–Stokes equations; the transverse viscosity either is the molecular viscosity or is exponentially increased close to the *outflow* to suppress spurious wiggles. A consistent Neumann pressure boundary condition is derived from the linear momentum equation; the latter is discretized in time using a high-order integration scheme in which the pressure equation is decoupled from the solution of the momentum equation, i.e. a projection technique is employed. Both Dirichlet and Neumann OBC's for the pressure field were tested.

4. Results

(a) BFS

In this case the authors predict *no steady state* but instead a time-periodic solution with a Strouhal number which they state was related to the instability of the shear layer emanating from the expansion. This result was in total disagreement with those of all the other contributors as well as the others in the minisymposium audience, though the authors point out that their results do agree with the earlier work of Kaiktsis *et al.*[18] However, in Reference 19 it has been shown that these spectral element solutions are spurious—the solution *is* a stable steady state.

(b) VSCC

Here the standard zero-stress boundary condition, i.e.

$$P = 2Re^{-1}\ \partial u_n/\partial n, \tag{30}$$

was first used to generate a solution which was in very good agreement with the benchmark solution. Then tests were performed using the new parabolized form of the equations and viscous sponges for both the Neumann and Dirichlet pressure OBCs. The results indicate that in general the parabolized boundary conditions perform better than the standard $\partial \mathbf{u}/\partial n = 0$ boundary condition for both Dirichlet and Neumann boundary conditions for the pressure at the outflow. In addition, a preliminary analysis of the effect of the exponentially increasing transverse viscosity at the outflow suggests that it has a stabilizing role on the spurious pressure modes that tend to propagate upstream and confines their influence to a thin sponge layer close to the outflow.

DISCUSSION

It is apparent from a perusal of the contributions that there is a myriad of techniques utilized by modellers to mediate the consequences of the unknown OBC. In the literature as well as in the contributions herein it is apparent that the finite difference community is a bit bolder in formulating and utilizing OBC's. There seems to be less variety in the finite element and spectral element areas, though the results here seem to indicate the *success rate* is reasonably good. Perhaps, it is a reflection of the basic rigidity in the fundamental discretization techniques, i.e. weighted residual vis-à-vis Taylor series. While it appears that oftentimes finite difference practitioners deal with boundary conditions as separate entities, there is less tendency for this in the weighted residual and, in particular, Galerkin techniques—part of the 'package deal' referred to in Reference 20.

In addressing the finite difference discretization issue, Strikwerda[21] states 'Many schemes also require additional boundary conditions, called numerical boundary conditions, to determine the solution uniquely', perhaps suggesting why finite difference techniques are oftentimes referred to as 'schemes' rather than 'methods'. The integral balance inherent in the weighted residual methods, on the other hand, intimately connects the bulk (domain) with the boundary; oftentimes this not only points to admissible boundary conditions but also provides discrete analogues—as in the case of *natural boundary conditions*. This coupling also necessarily engenders a mode of analysis and development which links the bulk balances and boundary conditions and abrogates the need for *numerical boundary conditions*. After some comments on the results of the test problems, which will necessarily be somewhat limited because of the small number of contributors, we will often offer some additional comments on issues associated with solvability of the discrete and semidiscrete versions of the Navier–Stokes (and advection–diffusion) equations.

For finite difference and finite volume discretizations the two most popular and sometimes successful OBC's were

$$\partial(\cdot)/\partial n = 0, \tag{31}$$

$$\partial(\cdot)/\partial t + \bar{u}_n\, \partial(\cdot)/\partial n = 0, \tag{32}$$

where $\bar{u}_n$ is usually some average velocity. These OBC's were generally successful for the unstratified test cases, primarily BFS, and less successful for the stratified ones, where, for example, in the PBCF noticeable distortions in the fields are present—albeit localized near the outlet. The quality of the result was somewhat dependent on the choice and implementation of $\bar{u}_n$ in the transient techniques. In the Galerkin finite element case the *natural* OBC's of normal and tangential forces set equal to zero were successful for both unstratified test cases and compared well with benchmark solutions. In the stratified test cases this boundary condition in general should be modified to account for at least pressure variation effects at the open boundary,[3] since in most cases the *viscous* normal force is comparatively small. The best way to modify this open boundary condition is still an open issue.

In Reference 22 the following question was asked: 'What is the implied BC for the pressure Poisson equation at an "outflow" (open) portion of the domain when the SIMPLE method (or one of its variants) is used to obtain a *solution* (i.e. Δx and $\Delta t \to 0$) of the Navier-Stokes equations?'. While this question seemingly remains open, it seems appropriate in this paper to at least hazard a guess as to its answer—if one exists. Thus below we propose at least one answer which, if not exact, may be close. We must hedge for two reasons: (i) there are very likely 'many' versions of SIMPLE and (ii) all descriptions of it that we have seen are unclear because of the complexity of the inner and outer iterations with respect to velocity–pressure coupling. The description offered below is placed in the related context of a simple (low-order) semi-implicit projection method that began with Chorin[23] (see also Reference 24). It was inspired principally by the work of Schutt,[1] but the recent paper by Blosch *et al.*[25] was also influential. Suppose the boundary comprises a Dirichlet portion (Γ_D) and an open portion (Γ_N). A two-step semi-implicit projection method for solving (1) and (2), where (1) is restated as

$$\partial \mathbf{u}/\partial t + \nabla p = \mathbf{f} + Re^{-1}\nabla^2 \mathbf{u}, \tag{33}$$

is as follows.

(i) Find an intermediate/provisional velocity ($\tilde{\mathbf{u}}$) by solving

$$(\tilde{\mathbf{u}} - \mathbf{u}_m)/\Delta t = \mathbf{f}_m + Re^{-1}\nabla^2 \tilde{\mathbf{u}} \quad \text{in } \Omega, \tag{34a}$$

with

$$\tilde{\mathbf{u}} = \mathbf{w}_{m+1} \quad \text{on } \Gamma_D, \qquad \partial \tilde{\mathbf{u}}/\partial n = \mathbf{0} \quad \text{on } \Gamma_N, \tag{34b,c}$$

where $\mathbf{w}(t)$ is given.

(ii) Find the final velocity ($\mathbf{u}_{m+1}$) from the projection

$$\mathbf{u}_{m+1} = \tilde{\mathbf{u}} - \Delta t \nabla P_{m+1} \quad \text{and} \quad \nabla \cdot \mathbf{u}_{m+1} = 0 \quad \text{in } \Omega, \tag{35a,b}$$

$$\mathbf{n} \cdot \mathbf{u}_{m+1} = \mathbf{n} \cdot \mathbf{w}_{m+1} \quad \text{on } \Gamma_D, \tag{35c}$$

$$\mathbf{n} \cdot \mathbf{u}_{m+1} = \mathbf{n} \cdot \tilde{\mathbf{u}} + \delta \quad \text{on } \Gamma_N, \tag{35d}$$

where δ (another 'intermediate' quantity) is derived from the overall (global) mass balance constraint, $\int_\Gamma \mathbf{n} \cdot \mathbf{u}_{m+1} = 0$, as

$$\int_{\Gamma_D} \mathbf{n} \cdot \mathbf{w}_{m+1} + \int_{\Gamma_N} (\mathbf{n} \cdot \tilde{\mathbf{u}} + \delta) = 0, \tag{36a}$$

i.e.

$$\delta = -\frac{1}{L_N}\left(\int_{\Gamma_D} \mathbf{n} \cdot \mathbf{w}_{m+1} + \int_{\Gamma_N} \mathbf{n} \cdot \tilde{\mathbf{u}}\right) = -\frac{1}{L_N}\int_\Gamma \mathbf{n} \cdot \tilde{\mathbf{u}}, \tag{36b}$$

where L_N is (in 2D) the length of Γ_N (L_N becomes the corresponding area, A_N, in 3D). The projection is realized by solving the following (consistent singular) Poisson equation implied by (35) and (36):

$$\nabla^2 P_{m+1} = \frac{\nabla \cdot \tilde{\mathbf{u}}}{\Delta t} \quad \text{in } \Omega, \tag{37a}$$

$$\partial P_{m+1}/\partial n = 0 \quad \text{on } \Gamma_D, \tag{37b}$$

$$\partial P_{m+1}/\partial n = -\delta/\Delta t \quad \text{on } \Gamma_N, \tag{37c}$$

after which (35a) yields $\mathbf{u}_{m+1}$. Thus, according to this simple interpretation of SIMPLE, the implied OBC for the PPE on Γ_N is simply the Neumann BC given by (37c).

Remarks

1. Global mass conservation is the only 'physics' behind this OBC.
2. A hydrostatic pressure mode always and legitimately exists, so that the induced matrix singularity may be legitimately removed by pegging one pressure.
3. Since $\tilde{\mathbf{u}} = \mathbf{u}_m + O(\Delta t)$ and $\int_\Gamma \mathbf{n} \cdot \mathbf{u}_m = 0$, it follows that $\delta = O(\Delta t)$ and thus $\partial P_{m+1}/\partial n = O(1)$ on Γ_N; the normal derivative is finite even for $\Delta t \to 0$.
4. Even though $\partial\tilde{\mathbf{u}}/\partial n = \mathbf{0}$ on Γ_N, $\partial \mathbf{u}_{m+1}/\partial n \neq \mathbf{0}$, but it is probably small—$O(\Delta t)$.
5. More than one open boundary (e.g. one pipe splitting into two or more) would cause trouble in that the same pressure gradient must be applied on each portion of Γ_N—thus indicating that the simple 'physics' (mass conservation) may not be enough in all cases.

Next we pursue further some additional aspects of the so-called 'free boundary condition' of Papanastasiou *et al.*[17] to show that it is sometimes a sort of *fuzzy boundary condition* (a new term that we coined during these OBC studies and one that should probably see widespread use) in that finite meshes often generate useful (even good) results but the $h \to 0$ limit leaves something to be desired related to boundary conditions *for the PDE's*. To this end, consider the simple scalar advection–diffusion equation in one dimension,

$$\partial T/\partial t + u\, \partial T/\partial x = K\, \partial^2 T/\partial x^2 + S, \quad 0 < x < L, \tag{38}$$

where u, K and S are constants and with, for simplicity, a boundary condition $T = 0$ at $x = 0$. The GFEM formulation of this is

$$\int \varphi_i \frac{\partial T^h}{\partial t} + u \int \varphi_i \frac{\partial T^h}{\partial x} + K \int \frac{\partial \varphi_i}{\partial x}\frac{\partial T^h}{\partial x} = \int \varphi_i S + \varphi_i K \frac{\partial T^h}{\partial x}\bigg|_{x=L}, \quad i = 1, 2, \ldots, N, \tag{39}$$

where $T^h(x, t) = \sum_{j=1}^{N} T_j(t)\varphi_j(x)$ and φ_i is the basis function associated with node i; $i = N$ is the last node, at $x = L$. Equation (39) is of course a coupled system of N ODE's; if the steady state version is the one of interest, $\partial T^h/\partial t$ is omitted and the finite-dimensional problem simplifies to that of a system of N simultaneous linear equations. In the conventional GFEM the boundary term (last term on right-hand side, obtained as a result of integration by parts of the diffusion term) is necessarily *replaced* by some relevant boundary condition information so that it, like the source term S, becomes data. For example, a general boundary condition is the Robin boundary, condition

$$K\, \partial T/\partial x + H(T - T_s) = q, \tag{40}$$

where H (heat transfer coefficient), T_s (sink temperature) and q (applied heat flux) are given data and the last term on the right-hand side of (39) is replaced by $\varphi_i[q - H(T^h - T_s)]|_{x=L}$; also, the term $\varphi_i H T^h|_{x=L}$ is transposed to the left-hand side. For the OBC case of interest herein we have $H = q = 0$ and the boundary condition $\partial T/\partial x = 0$ is automatically realized by simply dropping the last term on the right-hand side—sometimes referred to as a 'do nothing' (natural) boundary condition.

In the unconventional FEM of Papanastasiou *et al.*[17] the term $(\partial T^h/\partial x)|_{x=L}$ is treated quite differently. It is assumed *unknown* and thus transposed to the left-hand side, so that the weak formulation reads

$$\int \varphi_i \frac{\partial T^h}{\partial t} + u \int \varphi_i \frac{\partial T^h}{\partial x} + K\left(\int \frac{\partial \varphi_i}{\partial x}\frac{\partial T^h}{\partial x} - \varphi_i \frac{\partial T^h}{\partial x}\bigg|_{x=L}\right) = \int \varphi_i S, \quad i = 1, 2, \ldots, N, \tag{41}$$

and corresponds to their free boundary condition.

Let us compare the equations (39) with the term $(\partial T^h/\partial x)|_{x=L}$ dropped and (41) for two types of basis functions, linear and quadratic (both continuous piecewise polynomials). In the *linear* case the result is (for a uniform grid with node spacing h)

$$(h/6)(\dot{T}_{i-1} + 4\dot{T}_i + \dot{T}_{i+1}) + (u/2)(T_{i+1} - T_{i-1}) + (K/h)(-T_{i-1} + 2T_i - T_{i+1}) = Sh \qquad (42)$$

for $i = 1, 2, \ldots, N-1$, from (39) or (41), which clearly approximates (38). For $i =$ N, however, the results are different: (39) gives

$$(h/6)(2\dot{T}_N + \dot{T}_{N-1}) + (u/2)(T_N - T_{N-1}) + (K/h)(T_N - T_{N-1}) = Sh/2, \qquad (43)$$

whereas (41) results in

$$(h/6)(2\dot{T}_N + \dot{T}_{N-1}) + (u/2)(T_N - T_{N-1}) + 0 = Sh/2. \qquad (44)$$

While (43) clearly converges to $\partial T/\partial x = 0$ (the OBC) as $h \to 0$ and is the GFEM realization/approximation to $\partial T/\partial x = 0$ for finite h, (39) seems to yield, upon multiplication by $2/h$ and letting $h \to 0$,

$$\partial T/\partial t + u\, \partial T/\partial x = S \qquad (45)$$

which, if $S = 0$, is the popular 'radiation' OBC discussed earlier. Thus for $u \neq 0$ the free boundary condition seems to be legitimate and probably useful (at least when $S = 0$ and possibly otherwise). If, however, $u = 0$, the resulting transient heat equation seems to acquire the boundary condition $T = T_0(L) + St$, which is legitimate but again is probably only useful if $S = 0$. Finally, for the special limiting case of steady heat conduction, $K\mathrm{d}^2T/\mathrm{d}x^2 = S$, (44) becomes $Sh/2 = 0$ and the free boundary leads to an ill-posed problem in linear algebra. Again only $S = 0$ makes much sense.

For *quadratic* elements (basis functions) we obtain the following results.

Node N-1 from (39) or (41):

$$(4h/30)(\dot{T}_{N-2} + 8\dot{T}_{N-1} + \dot{T}_N) + (2u/3)(T_N - T_{N-2}) + (4K/3h)(-T_{N-2} + 2T_{N-1} - T_N) = (4h/3)S, \qquad (46)$$

which, upon division by $4h/3$, clearly approximates (38) as it should.

(ii) Node N from (39):

$$(h/15)(-\dot{T}_{N-2} + 2\dot{T}_{N-1} + 4\dot{T}_N) + (u/6)(T_{N-2} - 4T_{N-1} + 3T_N) + (K/6h)(T_{N-2} - 8T_{N-1} + 7T_N) = (h/3)S, \qquad (47)$$

which can be shown to approximate $\partial T/\partial x = 0$ as $h \to 0$.

(iii) Node N from (41):

$$(h/15)(-\dot{T}_{N-2} + 2\dot{T}_{N-1} + 4\dot{T}_N) + (u/6)(T_{N-2} - 4T_{N-1} + 3T_N) + (K/3h)(-T_{N-2} + 2T_{N-1} - T_N) = hS/3, \qquad (48)$$

which, upon division by $h/3$ and letting $h \to 0$, yields (38); i.e. in the limit the last nodal equation merely *replicates* the PDE and there seems to be no OBC at all. Furthermore, for the limiting case of steady heat conduction ($\dot{T}_i = 0$, $u = 0$) equation (48) is seen to be *idential* to (46), i.e. $K(-T_{N-2} + 2T_{N-1} - T_N)/h^2 = S$. While both are legitimate PDE approximations, it is the redundancy/repetition that is of concern. The last two rows of the resulting matrix are identical and thus the matrix is *singular*. For this simple case the singular system happens to be consistent—the right-hand-side vector is orthogonal to the null vector of the transpose matrix, whose null vector is $(0 \rightarrow 0\ \ 1\ \ -1)^{T}$. Thus a solution exists but is not unique. Any amount of the vector $(1\ \ 2\ \ 3\ \ \ldots\ \ N)^{T}$, which is the null vector of the heat conduction matrix (and mimics $T = x$, a function in the null space of d^2T/dx^2), can be added to any particular solution. Also, for S not constant, it is (much) worse: since the null vector is then generally not orthogonal to the right-hand-side vector, the system is ill-posed and has no solution.

These results would appear to generalize to the multidimensional energy and Navier–Stokes equations and, for the quadratic basis functions used by Papanastasiou *et al.*,[17] would be such that the PDE's themselves, rather than a PDE *boundary condition*, appear to be the $h \rightarrow 0$ limit of the so-called free boundary condition. Yet the results on two of the benchmarks presented by Papanastasiou *et al.*[17] are clearly quite good, even excellent, and this summarizes a particular case of a general dilemma—one which can occur with FDM or FEM: some fuzzy boundary conditions (numerical recipes) generated by clever researchers seem to deliver useful results on the *finite* meshes on which we all compute but which *appear* to be ill-posed in some sense (and for some problems) as $h \rightarrow 0$. We thus implore the mathematicians, especially those who are also numerical anlaysts, to analyse these fuzzy boundary conditions and try to make more sense out of what seems to us to be a still very confused area. We all need to learn more regarding 'finite h' mathematics and distinguish/separate it from $h \rightarrow 0$ analyses.

We conclude by mentioning one more FBC that often generates useful results in the FIDAP code.[26] In FIDAP 7.0 the following OBC is employed:

1. Start with the homogeneous NBC $\beta Re^{-1}\ \partial u_n/\partial n - P = 0$.
2. Obtain the solution of the discrete equations at the end of one iteration.
3. Update the NBC, this time inhomogeneous, in the normal momentum equation by assuming $\partial u_n/\partial n = 0$ and thus that $f_n = -P$, where P comes from the solution in step 2. This procedure leads to a force vector on the RHS.
4. Repeat until convergence; the 'convergence' is usually that involving a non-linear algebraic system. This procedure is called an FBC because we are not really sure which PDE BC is being approximated—although it appears to be $\partial u_n/\partial n = 0$, which seems to lose a pressure BC or even be ill-posed as $h \rightarrow 0$.

It appears that the mitigating factor for FBC's is *truncation error* associated with finite h; that is, if the discretized system is solvable, the non-uniqueness issue associated with the $h \rightarrow 0$ limit is 'resolved' by truncation error effects for finite h—the latter causing the solution to be 'unique' and to generate a 'truncation-error'-dependent solution branch. This being the case, one would expect the solution to experience 'problems' with mesh refinement, a property which has been demonstrated by Novy *et al.*[27] in modelling flow in porous media ('the quality of the solutions *decayed* with mesh refinement').

In closing, it may be relevant to report a very recent finding: at the (September 1993) Finite Elements in Fluids Conference in Barcelona, J. T. Oden announced the (re)discovery of the free boundary condition of Papanastasiou *et al.*[17] and showed some impressively accurate results for the BFS problem. T. J. Hughes then stated that he too has had some good experience with it, and attributed it to A. Mizukami, but avoided 'advertising' it because—as we have pointed out—'it seems like no boundary condition at all'.

Finally, the varied and still confused state of OBC theory and application is illustrated by the recent publications of Johansson[28] and Jin and Braza.[29] The former utilized homogeneous higher-order derivatives on the velocity components and specified the pressure, while the latter imposed an advection–diffusion equation on the velocity components and a homogeneous third derivative on the 'pressure' variable. For another FBC see the recent publication of Ramaswamy,[30] who implemented the OBC designed by Shimura and Kawahara.[31]

CONCLUSIONS

We have made some attempts at shedding more light on the difficult and unresolved area of seeking good OBCs for incompressible flow simulations. It has been an exercise in frustration and we are not thrilled with the results obtained, even though they may still be useful to some researchers; thus we pass the baton. We believe that the most important issue for incompressible flows is that the incompressibility constraint is all-pervasive and even shows up (or should) on open boundaries, with the concomitant (and often awkward) result of coupling the pressure and the normal velocity there—either explicitly or implicitly (see also Reference 32). We have also identified what we term fuzzy boundary conditions that are begging for deeper understanding.

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